Many-Valued Models

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Abstract

Many-valued models, besides providing a natural semantical interpretation for several non-classical logics, constitute a very sharp tool for investigating and understanding meta-logical properties in general. Although open to debates from the philosophical perspective, seen from the mathematical viewpoint many-valued matrices and algebras are perfectly well-defined mathematical objects with several attractive properties. This tutorial intends to review the main results, techniques and methods concerning the application of the many-valued approach to logic as a whole.

1 On many-valued thinking

The technique of using finite models defined by means of tables (which turns out to be finite algebras) is arguably older than many-valued logics themselves, and has provided much information not only about non-classical systems as relevant logics, linear logic, intuitionistic logics and paraconsistent logics, but also about fragments of classical logic. The problem of enumerating such algebras satisfying given constraints is an interesting general problem and has received attention from several different areas.

In this tutorial we present an elementary but general approach on small finite models, showing their relevance and reviewing some elementary methods and techniques on their uses.

There are many significant names in the history of logic that are connected with the idea of many-valuedness, for different reasons. The Polish logician and philosopher Jan Łukasiewicz was born in Łvov. His philosophical work developed around on mathematical logic; Łukasiewicz dedicated much attention to many-valued logics, including his own hierarchy of many-valued propositional calculus, considered to be the first non-classical logical calculus. He is also responsible for an elegant axiomatizations of classical propositional logic; it has just three axioms and is one of the most used axiomatizations today:

$(P^1\text{-}1)\quad A \to (B \to A)$

$(P^1\text{-}2)\quad (A \to (B \to C)) \to ((A \to B) \to (A \to C))$

$(P^1\text{-}3)\quad (\neg A \to \neg B) \to (B \to A)$

and *modus ponens* is the only rule.

Although we may credit Łukasiewicz for having introduced a third truth-value in Philosophy, relevant contribution of truth -values in Mathematics should be credited to Paul Bernays and Emil Post, although part of the material produced in the Hilbert school of Göttingen around 1920 (notably Bernays's work) remained unpublished (cf. [Zac99]).

The method of using many-valued matrices for independence proofs, a central point in this tutorial, was proposed for the first time by Bernays in his *Habilitationsschrift* (Bernays 1918), and was also discovered independently by Łukasiewicz and Tarski. Among his several contributions to logic, Bernays introduced the first three- and four-valued models. Bernays's approach to proving independence of the axioms involved methods that can be called many-valued logics. For learning more about his role in this aspect see [Zac99] and [Pec95].

For instance, Heyting (1930) took the ideas of Bernays into consideration when proving the independence of his axiom system for intuitionistic logic, and Gödel (1932) was influenced by such ideas when he defined a sequence of sentences independent of intuitionistic propositional calculus; the resulting many-valued logics are now known as Gödel logics.

The many-valued approach is also important for distinguishing the cases where *no* finite-valued semantical interpretation is possible. This is the case of the work of Gödel (cf.[G3̈2]) in proving that there is no finite many-valued semantics for intuitionistic logic, and of James Dugundji (cf. [Dug40]) in proving that there is no finite many-valued semantics for modal logics. More recently, this was also the case (cf. [CCM]) of proofs that several paraconsistent logics are also uncharacterizable by finitely-valued semantics.

Along with Łukasiewicz and Bernays, Charles Pierce and Emil Post are usually credited with the truth-table method for determining propositional validity. Even Ludwig Wittgenstein is credited for having the idea of organizing the truth-values in table format!

Emil Leon Post was a Polish-American mathematician and logician born in a Jewish family in Augustow,Poland, and died in New York City, USA.

In 1936 Post proposed an abstract computer model now called "Post machines", independently of Alan Turing's model known as " Turing machines".

Emil L. Post's dissertation of 1920 provided metatheoretical results about the propositional calculus. It contains an explicit account of the truth table method, Among them are, for instance, that the truth table method provides a decision procedure for derivability. Post's paper contains a number of other contributions. These are, on the one hand, a discussion of truth-functional completeness, and on the other, an independent introduction of many-valued logics.

There are various reasons one may desire to introduce more than two values:

- Reasoning with truth-value gaps

  In 1938 Ł ukasiewicz delivered a lecture to the Circle of Scientists in Warsaw, Genesis of three-valued logic. Ł ukasiewicz considered the discovery of many-valued logics as important as of non-Euclidean geometry, and thought that they make possible “other ways of speaking of reality”. The fundamental idea in the birth of three-valued logic was adding a third value to the matrix of bivalued logic, having in mind an intuitive interpretation of this new value.

  The interpretation Ł ukasiewicz had in mind was linked with Aristotles *Perihermeneias* and sentences on future contingent facts, that were in his view neither true nor false. Future contingents. Aristotle raised the possibility that sentences about the future are not currently either true or false.

  Sentences with false presuppositions, such as ”The present king of France is bald” can be considered as neither true nor false; on may consider the they have rather a false presupposition, as it is the case that France has presently no kings. However, it may be interesting to treat these cases by having a third truth-value for ”neither true nor false”.

- Reasoning with truth-value gluts

  We may also think about the possibility that some sentences can be both true and false, as it is the case of some paradoxes. Consider the following version of the ”liar” paradox: “This sentence is false” . Suppose that it is true. Then, since it says it is false, it must be false. But, on the other hand, if it is false, then what it says (namely that it is false) is true. So it is true if and only if it is false. A possible reaction to a paradox like this is to add a third truth-value for “both true and false”.

  A philosophical application of three-valued logics to the discussion of paradoxes was proposed by the Russian logician Bochvar already in 1939 (cf. [Boc39]), and a useful mathematical application to partial computable functions and relations by the North-American logician Stephen Cole Kleene in his famous book [Kle50]. Later Kleene’s connectives also became philosophically interesting as a technical tool to determine fixed points in the revision theory of truth initiated by Saul Kripke in 1975.

- Reasoning with epistemological possibilities

  Sometimes we may want to add a third and fourth truth values for “unknown”, so that the three truth-values would be “known to be true,” “known to be false”, “unknown” or “neither”, and “ both”. This is the case, for example, of the well-known case of the four-valued semantics introduced by Nuel Belnap in [Bel77] for expressing deductive processes connected to databases It is well-known that databases may contain explicit or implicit contradictions, which may come from equally reliable sources. The use of a classical deductive process would not be appropriate in the presence of a contradiction, since any arbitrary information would classically derivable. These four-valued possibilities would explain “how a computer should think”, and constitute a first paraconsistent approach to databases, an active area of research nowadays (see also [CMdA00]).

This new way of looking to logico-philosophical scenario was not free of discussion, however. Stanisław Lesniewski argued that a third logical value never appears in scientific argumentation, and considered the third value as no sense, because "no one had been able until now to give to the symbol 2 introduced in a three-valued matrix any intelligible sense, which may ground this or that realistic interpretation of this logic". .

This type of criticism, however, blaming pure science for the lack of use of many-valuedness may easily lose force when new scientific objects come into light. The Princeton University Bicentennial Conference on Problems of Mathematics took place in December, 1946, and was the first major international conference gathering logicians and mathematicians after World War II. The session on Mathematical Logic had the participation of Alonzo Church, Alfred Tarski and Alan Rosser, among other big names. Rosser gave a talk on the possibilities of Birkhoff and von Neumann suggestions of using many-valued logics as foundation for quantum mechanics, what is the subject of a strong research effort today. What this shows is that, independent of any intrinsic many-valuedness character attached to scientific objects, it may be *convenient* to look to such objects from the many-valued viewpoint.

This tutorial is not only about many-valued logics, but about the uses of the many-valued approach to mathematical objects and concepts.

Ł ukasiewicz, in 1929, recognized the important role of many-valued models in logic: "Actually, it is the method of proving the independence of propositions in the theory of deduction which has occasioned our research into many-valued logics" (cf. [Lu29]).

We are interested here in this basic role of many-valuedness in providing new theoretical models, as the ones used in independence proofs which usually use logical matrices with more than two truth-values. In such cases, we will not be necessarily interested is in an intuitive understanding of the truth-values, but in their mathematical role of providing new, not-yet-thought, possible interpretations of a theory.

In this sense, such models test the limits of the theories, and constitute beautiful models with rich potential.

This tutorial is organized into the following topics:

- On many-valued thinking
- Some three-valued Logics
- Independence of axioms of **CPL**
- An incapacity of two-valued and three-valued models
- The system **Q** of Mostowski-Robinson: models for Arithmetic
- Proving the weakness of the system ***Q***
- Tarski's High School Problem and exotic identities
- The Finite Basis Problem and weird small models
- Why modal logics are not many-valued

- Why intuitionistic logic is not many-valued
- Why certain paraconsistent logics are not many-valued
- Possible-translations semantics

2 Some three-valued logics

Instead of starting with classical logic, we will start by discussing some many-valued logics. As examples of many-valued models, we show here some examples of three-valued tables that characterize some important three-valued logics. Although there are literally hundreds of three-valued logics (see for example [Got01]),we just treat here four particular cases: the historically relevant Łukasiewicz's and Gödel's three-valued logics, the paraconsistent three-valued system P^1 and the paracomplete (or weakly-intuitionistic) three-valued system I^1 .

By a *signature*we mean a collection of logical operators (connectives). Given a logic **L** defined by a set of axioms and rules, $\Gamma \vdash_{\mathbf{L}} \alpha$ means, in general, that there is proof in **L** of α from the premises in Γ. The subscript may be omitted when obvious from the context. If Γ is empty we say that α is a *theorem*.

The propositional three-valued logic known as L_3 was first proposed by Jan Ł ukasiewicz in 1920 (though his results were published later, by that time he was already concerned with use of models to show consistency, cf. [Lu03]). An axiomatization of L_3 was given by M. Wajsberg in 1931 by using implication and negation as primitive connectives (Alan Rose proposed in 1951 several other alternative axiomatizations for L_3):

(L-1) $A \to (B \to A)$

(L-2) $(A \to B) \to (B \to C) \to (A \to C)$

(L-3) $(\neg A \to \neg B) \to (B \to A)$

(L-4) $(A \to \neg B) \to B) \to B)$

The following rules of inference were also assumed:

Substitution: Any well-formed formula may be substituted for a propositional variable in all its occurrences in a theorem or axiom.

Modus Ponens (MP): If A and $A \to B$ are theorems, then B is also a theorem.

In the famous Polish (prefix) notation, introduced by the same Łukasiewicz, these axioms are written as $CpCqp$, $CCpqCCqrCpr$, $CCNpNqCqp$ and $CCCpNppp$.

The following matrices, reading the truth-values T, U and F as "true", "undetermined" and "false" and considering only T as a distinguished value, give a sound and complete semantical interpretation for L_3, in the sense that all theorems receive value T,and no other formulas receive value T.

	T	U	F
$\overset{P^1}{\neg}$	F	U	T

$\overset{P^1}{\to}$	T	U	F
T	T	U	F
U	T	T	U
F	T	T	T

The three-valued system P^1 was introduced in [Set73] in order to obtain the simplest possible paraconsistent calculus. P^1 is a subsystem of CPC, and is maximal in the sense that adding to its axioms any classical tautology which is not a P^1-tautology the resulting system collapses to CPC.

Axiomatically, P^1 is characterized in the following way, using the language of CPC:[1]

(P^1-1) $A \to (B \to A)$

(P^1-2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(P^1-3) $(\neg A \to \neg B) \to ((\neg A \to \neg\neg B) \to A)$

(P^1-4) $(A \to B) \to \neg\neg(A \to B)$

and *Modus Ponens* is the only rule. It can be proved (cf. [Set73]) to be complete with respect to the following matrices, where $\to$ and $\neg$ are primitive, and $\wedge$ and $\vee$ are defined. The truth values are T, T^*, F, of which T, T^* are distinguished. Intuitively, T and F mean plain truth and falsity, whereas T^* can be understood as "truth by default", or "by lack of evidence to the contrary".

	T	T^*	F
$\overset{P^1}{\neg}$	F	T	T

$\overset{P^1}{\to}$	T	T^*	F
T	T	T	F
T^*	T	T	F
F	T	T	T

The primitive negation of $\mathbf{P}^1$ is paraconsistent (thus, weak with respect to implication) in the sense that, for example, $A \to (\neg A \to B)$ is not a P^1 tautology, as can easily be checked from the given matrices assigning the truth-value T^* to A and F to B. It is possible, however, to define in P^1 a *strong negation* $\sim A$ which recovers the full power of classical negation: $\sim A =_{def} \neg(\neg A \to A)$, which gives the following table:

	T	T^*	F
$\sim$	F	F	T

Using the strong negation, we can also define conjunction $A \overset{P^1}{\wedge} B$ and disjunction $A \overset{P^1}{\vee} B$ in P^1 as follows:

$$A \overset{P^1}{\wedge} B =_{def} \neg(A \to \sim B)$$

$$A \overset{P^1}{\vee} B =_{def} (\sim A \to B)$$

$\overset{P^1}{\wedge}$	T	T^*	F
T	T	T	F
T^*	T	T	F
F	F	F	F

$\overset{P^1}{\vee}$	T	T^*	F
T	T	T	T
T^*	T	T	T
F	T	T	F

[1] In the formulation of [Set73] there exists an additional axiom which can be deduced from the ones given here.

The truth-table of the consistency connective $\circ$ can be defined in $\mathbf{P}^1$ by $\circ\alpha \stackrel{\text{def}}{=} \neg\neg\alpha \vee \neg(\alpha \wedge \alpha)$. The logic $\mathbf{P}^1$ is controllably explosive with respect to arbitrary non-atomic formulas in the sense that the paraconsistent behavior obtains only for atomic formulas: $\alpha, \neg\alpha \vDash \beta$, for arbitrary non-atomic α. Moreover, $\vDash \circ\alpha$ also holds for non-atomic α.

The system I^1 was introduced in [SC95] as a three-valued dual counterpart of the system P^1. The truth values of I^1 are T, F^*, F, of which only T is distinguished. Intuitively, again T and F mean plain truth and falsity, whereas F^* can be understood as "false by default", or "by lack of positive evidence".

The system $\mathbf{I}^1$, instead of paraconsistent, possess an intuitionistic character, in the sense that, for example, $\neg\neg A \to A$ is not an $\mathbf{I}^1$ tautology, as can be checked from the matrices below, assigning the truth-value F^* to A. Moreover, in $\mathbf{I}^1$ all the axioms of the well-known Heyting system for intuitionistic logic are valid, and the law of excluded middle is not valid (for the disjunction defined below).

The axioms of I^1 (in the same language of CPC, having *Modus Ponens* as the only rule) are:

Il-1) $A \to (B \to A)$

(Il-2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(Il-3) $(\neg\neg A \to \neg B) \to ((\neg\neg A \to B) \to \neg A)$

(Il-4) $\neg\neg(A \to B) \to (A \to B)$

and it can be shown (cf. [SC95]) to be complete with respect to the matrices below, where $\to$ and $\neg$ are primitive connectives. As mentioned before, the truth values are T, F^*, F, and T is the only distinguished value:

	T	F^*	F
$\overset{I^1}{\neg}$	F	F	T

$\overset{I^1}{\to}$	T	F^*	F
T	T	F	F
F^*	T	T	T
F	T	T	T

I^1 can be proved to be maximal in a sense similar to the case of P^1, and it is possible to define in I^1 a *weak negation* $\simeq A$ which has all the properties of classical negation: $\tilde{\neg} A =_{def} A \to \neg A$ giving the following table:

	T	F^*	F
$\simeq$	F	T	T

We can also define conjunction $A \overset{I^1}{\wedge} B$ and disjunction $A \overset{I^1}{\vee} B$ for this system in the following way:

$$A \overset{P^1}{\wedge} B \quad =_{def} \quad \neg(A \to \simeq B)$$

$$A \overset{P^1}{\vee} B \quad =_{def} \quad (\simeq A \to B)$$

I^1 $\wedge$	T	F^*	F
T	T	F	F
F^*	F	F	F
F	F	F	F

I^1 $\vee$	T	F^*	F
T	T	T	T
F^*	T	F	F
F	T	F	F

The notion of society semantics as a way to provide new semantical interpretations for many-valued logics by means of collections of two-valued semantics was introduced in [CLM99]. It can be shown that, under certain conditions, the behaviour of certain biassertive societies (i.e., societies with two agents) is essentially equivalent to three-valued logics. In particular, closed biassertive societies are equivalent to I^1, and open biassertive societies are equivalent to P^1. For details, see [CLM99].

3 Independence of axioms of CPL

Let the signature Σ^+ denote the signature Σ without the symbol $\neg$, and For^+ be the corresponding formulas; Positive classical logic **CPL**$^+$ can be axiomatized in the signature Σ^+ by the following axioms and (MP):

Positive classical logic CPL$^+$
Axiom schemas:

(Ax1) $\alpha \to (\beta \to \alpha)$

(Ax2) $(\alpha \to \beta) \to ((\alpha \to (\beta \to \gamma)) \to (\alpha \to \gamma))$

(Ax3) $\alpha \to (\beta \to (\alpha \wedge \beta))$

(Ax4) $(\alpha \wedge \beta) \to \alpha$

(Ax5) $(\alpha \wedge \beta) \to \beta$

(Ax6) $\alpha \to (\alpha \vee \beta)$

(Ax7) $\beta \to (\alpha \vee \beta)$

(Ax8) $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$

(Ax9) $\alpha \vee (\alpha \to \beta)$

Inference rule:

(MP) $\dfrac{\alpha,\ \alpha \to \beta}{\beta}$

Classical Propositional Logic **CPL**is obtained from **CPL**$^+$ by adding two controversial axioms: the law of excluded middle:

(exc) $\alpha \vee \neg\alpha$.

, and the law of explosion:

(exp) $\alpha \to (\neg\alpha \to \beta)$

).

The following short axiomatization of **CPL** based on negation and implication, with *Modus Ponens* as the only inference rule is given in [Eps00], Chapter II.L (it was also considered before, called Bigos-Kalmár axioms in [Fla78]):

Short axiomatization

Modus ponens (**MP**): $\dfrac{A, \quad A \to B}{B}$

plus the following schemas:

(1) $\neg A \to (A \to B)$

(2) $A \to (B \to A)$

(3) $(A \to B) \to ((\neg A \to B) \to B)$

(4) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

A recurrent problem is to show that the axioms are independent using the simplest possible matrices. In [Eps00], Chapter VIII.G, some four-valued matrices are used to prove independence of these axioms. It should be noted however that the model proposed for proving independence of Axiom (2) is wrong: indeed, the suggested table does not respect modus ponens [2]. Moreover, the independence of Axiom (1) and Axiom (3) uses the unnecessarily complicated four-valued models.

We show here that it is possible to employ two- and three-valued models to show independence of Axioms (1), (2) and (3), but no three-valued model can separate Axiom (4).

I. Independence of Axiom (1): $\neg A \to (A \to B)$

Consider the tables for implication and negation given by the three-valued paraconsistent calculus P^1 (introduced in [Set73]), where 0 and 1 are distinguished (designated) truth-values:

$\to$	0	1	2
0	0	0	2
1	0	0	2
2	0	0	0

	$\neg$
0	2
1	0
2	0

Axioms (2) and (4) are also axioms of P^1.

It is easy to check that Axiom (1) $\neg A \to (A \to B)$ is also validated.

To see that Axiom (1) can be assigned value 2, just assign value 1 to A and 2 to B (in shorthand, $\neg 1 \to (1 \to 2) = 0 \to (1 \to 2) = 2$).

[2]This observation is due to Rodrigo Freitas at CLE- UNICAMP

Is it possible to employ smaller (in this case, two-valued) models? For this axiom, the answer is positive: we will reproduce some two-valued models given in [Fla78], where 0 as the distinguished value:

$\to$	0	1
0	0	1
1	0	0

	$\neg$
0	0
1	0

II. Independence of Axiom (2): $A \to (B \to A)$

Consider the tables for implication [3] and negation given below, where 0 and 1 are distinguished truth-values:

$\to$	0	1	2
0	1	2	2
1	1	1	2
2	1	1	1

	$\neg$
0	2
1	2
2	1

Again axioms (1) , (3) and (4) are validated by these tables, while Axiom (2) can be assigned value 2: just assign value 0 to A and 2 to B (in shorthand, $0 \to (2 \to 0) = 0 \to 1 = 2$). Is it possible in this case, to simplify the models? We do not know the answer; we *conjecture* that no two=valueid models is able to show independence of axiom Axiom (2).

III. Independence of Axiom (3): $(A \to B) \to ((\neg A \to B) \to B)$

Consider the tables for implication and negation given by the three-valued paracomplete calculus I^1 (introduced in [SC95]), where 0 is the only distinguished truth-value:

$\to$	0	1	2
0	0	2	2
1	0	0	0
2	0	0	0

	$\neg$
0	2
1	2
2	0

Again axioms (2) and (4) are also axioms of I^1. It is easy to check that Axiom $\neg A \to (A \to B)$ is also validated by I^1 tables. To see that Axiom (3) can be assigned value 2, just assign value 1 to A and 2 to B (in shorthand, $(1 \to 2) \to ((\neg 1 \to 2) \to 2) = (1 \to 2) \to ((2 \to 2) \to 2) = 2$).

Again, in this case the models can be simplified to two-valued, taking 0 as the distinguished value (cf.[Fla78]):

$\to$	0	1
0	0	1
1	0	0

	$\neg$
0	1
1	1

[3] Tables obtained with the help of the program MaGIC: ("Matrix Generator for Implication Connectives"). Thanks to John Slaney

IV. Independence of Axiom (4): $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$ We use the four -valued tables given in [Eps00] Chapter VIII.G (due to George Hughes):

$\to$	1	2	3	4
1	1	2	3	4
2	1	1	3	1
3	1	2	1	2
4	1	1	1	1

	$\neg$
1	4
2	3
3	2
4	1

Again axioms (1), (3) and (4) are validated by these tables, while Axiom (2) can be assigned value 2: just assign value 2 to A, 4 to B and 3 to C (in shorthand, $(2 \to (4 \to 3)) \to ((2 \to 4) \to (2 \to 3)) = 3$).

In this case, however, no model with less than four truth-values is able to guarantee independence of Axiom (4), as shown in next section.

4 An incapacity of two-valued and three-valued models

As we saw in the previous section, Axioms (1), (2) and (3) can be proven to be independent by means of three-valued models, and (4) by means of a four-valued modus ponens. An interesting question is the following: is it possible to use a three-valued matrix instead? I show below that this is impossible: no three-valued model is sufficient to show the independence of Axiom (4):

$$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$$

I first show some consequences obtained from Axioms (1), (2) , (3) and (**MP**) which will be helpful to establish the conditions for any possible three-valued model. **L** be the subclassical logic defined by Axioms (1), (2) , (3) and (**MP**); $\vdash A$ indicates that A is a theorem of **L** , and $A \dashv\vdash B$ indicated that A and B are mutually derivable in **L**.

Proposition 4.1. **(a)** $\vdash A \to A$

(b) $\vdash A \to \neg\neg A$

(c) $A \to (A \to B) \vdash (A \to B)$

Proof. For (a):

1. $\neg A \to (A \to A)$ [Ax 1]
2. $A \to (A \to A)$ [Ax 2]
3. $(A \to (A \to A)) \to ((\neg A \to (A \to A)) \to (A \to A))$ [Ax 3]
4. $\top \to (\varphi \to \top)$ [MP 2, 3]
5. $A \to A$ [MP 1, 4]

For (b):

1. $\neg A \to (A \to \neg\neg A)$ [Ax 1]
2. $\neg\neg A \to (A \to \neg\neg A)$ [Ax 2]
3. $\neg A \to (A \to \neg\neg A) \to [(\neg\neg A \to (A \to \neg\neg A)) \to (A \to \neg\neg A)]$ [Ax 3]
4. $((\neg\neg A \to (A \to \neg\neg A)) \to (A \to \neg\neg A))$ [MP 2, 3]
5. $A \to \neg\neg A$ [MP 1, 4]

For (c):

1.	$A \to (A \to B)$	[Hyp.]
2.	$\neg A \to (A \to B)$	[Ax 1]
3.	$A \to (A \to B) \to [(\neg A \to (A \to B)) \to (A \to B)]$	[Ax 3]
4.	$[(\neg A \to (A \to B)) \to (A \to B)]$	[MP 1, 3]
5.	$A \to B$	[MP 2, 4]

□

Let M be any model for $\boldsymbol{L}$ interpreting $\to$ and $\neg$, [4] let D and ND be the (non-empty) sets of, respectively, distinguished and non-distinguished truth-values of M. Then the following properties hold (where d, d' are any distinguished truth-values, and n, n' are any non-distinguished truth-values, and x is any truth-value):

Proposition 4.2. *Let M be any model for $\mathbf{L}$ of any cardinality, then:*

(a) $x \to d \in D$

(b) $d \to n \in ND$

(c) $x \in D$ *iff* $\neg\neg x \in D$

Proof. For (a): Axiom (2) guarantees that $d \to (x \to d) \in D$; by MP, $x \to d \in D$.

For (b): If $d \to n \in D$, by MP $n \in D$, absurd.

For (c): If $x \in D$ then $\neg\neg x \in D$ by Proposition 4.1 (b).

Conversely, if $\neg\neg x \in D$, since $\neg\neg x \to (\neg x \to x)$ (by Axiom (1)), it follows that $(\neg x \to x) \in D$. Also, $(x \to x) \in D$. (by Proposition 4.1), but by Axiom (3) $(x \to x) \to ((\neg x \to x) \to x) \in D$. and by MP $x \in D$. □

Proposition 4.3. *Let M be a three-valued model for $\mathbf{L}$, then $x \in D$ iff $\neg x \in ND$.*

Proof. From to right: suppose that $x \in D$ and $\neg x \in D$ for some x; then by Axiom (1), $\neg x \to (x \to n) \in D$, and consequently, by MP, $n \in D$, absurd.

The converse requires a more involved combinatorial argument. Suppose that, for some x, $x \in ND$ and $\neg x \in ND$. The argument has to be divided into two cases:

Case 1: Suppose $|D| = 2$. Then there is just one x in ND, and as by hypothesis $x = \neg x = n \in ND$, we obtain $(x \to x) = (\neg x \to x) \in D$ by Proposition 4.1, and thus, since by Axiom (3) $(x \to x) \to ((\neg x \to x) \to x) \in D$, then $x \in D$, absurd.

Case 2: Suppose $|D| = 1$. In this case, by the left-to-right part above, $\neg d \in D$ and $ND = \{n, n'\}$. Suppose $\neg d = n$: I will show that $\neg n = n' = d$. Indeed, $\neg n \neq n$, for otherwise we get the same contradiction as in Case 1. So, if $\neg n \neq d$, then $\neg n = n'$, and thus $\neg\neg d = \neg n = n'$, a contradiction with item (c) above. Hence $\neg n = d$.

It remains to be shown that $\neg n' = d$. If not, since again $\neg n' \neq n'$ for the same reason as above, then $\neg n' = n$. But in this case $\neg\neg n' = \neg n = d$, again a contradiction with item (c) above. Hence $\neg n' = d$.

[4] As usual, we are employing the same symbols $\to$ and $\neg$ for the connectives and their interpretation

Therefore in any cases if $x \in ND$ then $\neg x \in D$ what establishes the converse. □

I now show that such conditions eliminate all possible three-valued models that can falsify Axiom (4) while satisfying the other axioms.

Proposition 4.4. *Any two-valued or three-valued models that satisfy Axioms (1)-(3) also satisfies the schema* $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

Proof. It is almost immediate to see that any two-valued models will be excluded as showing the Independence of the schema $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$. Indeed, in any two-valued models the truth-values of A and B, A and C or B and C will coincide.

If the truth-values of A and B coincide, it is impossible to disprove the schema $(A \to (A \to C)) \to ((A \to A) \to (A \to C))$, since Proposition 4.2 implies that this would only be possible if $(A \to (A \to C))$ is distinguished and $((A \to A) \to (A \to C))$ is not, but by Proposition 4.1 (c) and Axiom (2) $A \to (A \to B) \vdash (A \to A) \to (A \to C)$, a contradiction.

If the truth-values of A and C coincide, it is impossible to disprove the schema $(A \to (B \to A)) \to ((A \to B) \to (A \to A))$, as a consequence of Proposition 4.1 (c) and Axiom (2). If the truth-values of B and C coincide, it is impossible to disprove the schema $(A \to (B \to B)) \to ((A \to B) \to (A \to B))$, again as a consequence of Proposition 4.1 (c) and Axiom (2). This takes care of the two-valued cases.

Suppose now that M is a possible three-valued model for Axiom (1), (2) and (3), and closed under MP, such that M falsifies Axiom (4). In M, the table for will have the following structure (under convenient permutation of rows and lines, and where entries D and ND in the tables mean any, respectively, distinguished or non-distinguished truth-value):

Case 1: $|D| = 2$

As a consequence of Proposition 4.2 (a), (b) and (c), and noticing that $n \to n \in D$ by Proposition 4.1, we have:

$\to$	d	d'	n
d	D	D	n
d'	D	D	n
n	D	D	D

Case 2: $|D| = 1$

As a consequence of Proposition 4.2 and Proposition 4.3, and again noticing that $n \to n \in D$ and $n' \to n' \in D$ by Proposition 4.1, we have:

$\to$	d	n'	n
d	d	ND	ND
n'	d	d	?
n	d	?	d

It is clear that $n' \to n = d$ and $n \to n' = d$. Indeed, suppose that $n' \to n \in ND$. By Axiom (1) $\neg n' \to (n' \to n) \in D$. Also, by Proposition 4.3 $\neg n' \in D$, and thus by Proposition 4.2 (c) $\neg n' \to (n' \to n) \in ND$, a contradiction. A similar argument shows that $n \to n' = d$.

The schematic table can now be determined as:

$\to$	d	n'	n
d	d	ND	ND
n'	d	d	d
n	d	d	d

It remains to be shown that neither the first nor the second schematic tables can falsify Axiom (4): Indeed.

(i) By hypothesis, $(A \to (B \to C)) \to ((A \to B) \to (A \to C)) \in ND$;

(ii) From (i) and the table, $(A \to (B \to C)) \in D$, $(A \to B) \to (A \to C) \in ND$;

(iii) From $(A \to B) \to (A \to C) \in ND$, (ii) and the table, $(A \to B) \in D$, $(A \to C) \in ND$;

(iv) From $(A \to C) \in ND$ (iii) and the table, $A \in D$, $C \in ND$;

(v) From $A \in D$ and $(A \to B) \in D$ it follows by MP that $B \in D$

(vi) From $B \in D$ (v) and $C \in ND$ (iv) it follows from Proposition 4.2 (c) that $(B \to C) \in ND$, From $A \in D$ (iv) and $(B \to C) \in ND$ (vi) it follows from Proposition 4.2 (c) that $(A \to (B \to C)) \in ND$, absurd (as it contradicts (ii)).

Hence there are no three-valued matrices that can falsify Axiom (4) . □

The same techniques of this section can be adapted to study the relative incapacitiy of other many-valued models. Although some well-known results of Gödel, Dugundji, Dummett and Harrop show that there are many decidable propositional logics (even subsystems of classical logic) that cannot be characterized by finite matrices, an independent interesting problem is the following: Is there a sequence $\boldsymbol{L}_k$ of propositional logics such that, for each n, there exists $\boldsymbol{L}$ in $\boldsymbol{L}_k$ such that the independence of axioms in $\boldsymbol{L}$ cannot be characterized by n-valued finite matrices, but can be by $(n+1)$-valued finite matrices?

We conjecture that this is true; the interest of this type of property is that, in such cases, the universe of n-valued finite matrices (when closed under certain conditions) has some intrinsic hidden structures, as Proposition4.4 reveals for the case of 3-valued finite matrices.

5 The system *Q* of Mostowski-Robinson: models for Arithmetic

If we take as axioms the inductive definitions of addition and multiplication and the ones that say that successor (on non-zero numbers) is a 1-1 function, we obtain the axiom system $\boldsymbol{Q}$ for first-order arithmetic. The arithmetic axioms of $\boldsymbol{Q}$ are due to Raphael Robinson, in 1950; for the history see [TMR53],p.39.

The partial recursive functions are the smallest class of functions containing zero, successor, the projections, addition, multiplication, and the characteristic function for equality and closed under composition and the μ-operator. It is well-known that all the recursive functions are representable in $\boldsymbol{Q}$ and the representable functions are closed under composition and the μ-operator. This makes possible to translate the numerical versions of assertions about mathematical systems containing a certain amount of artithmetic back into the system itself, what is an essential ingredient for proving the celebrated Theorems of Gdel.

The system of $\boldsymbol{Q}$ plays an extremely important role in logic and model theory. Because the recursive functions are representable in the formal system $\boldsymbol{Q}$, we obtain that the set of theorems of $\boldsymbol{Q}$ is undecidable. Moreover, by adding the axiom schema of induction to $\boldsymbol{Q}$, we obtain Peano Arithemetic $\boldsymbol{PA}$, and the Second Incompletenss Theorem of Gdel shows that $\boldsymbol{PA}$ cannot prove its own consistency.

In [EC00], chapters 21 and 22, it is carefully shown that all the recursive functions can be represented using the proof machinery of $\boldsymbol{Q}$, and chapters 23 and 24 treat the Theorems of Gdel in full detail, what is not our objective here: our objective is to show how, using finitely-presented methods, we can show what *cannot* be proved in $\boldsymbol{Q}$.

Alfred Tarski, Andrzej Mostowski, and Raphael Robinson generalized the ideas underlying $\boldsymbol{Q}$, developing some techniques for showing that several theories are undecidable. This is known in the literature as 'Tarski-Mostowski-Robinson theorem', but it is also recognized that Myhill and Bernays had previiysly contributed to it. The Tarski-Mostowski-Robinson results can be used show the undecidability of several important algebraic theories, as the elementary theories of rings, commutative rings, integral domains, ordered rings, ordered commutative rings, and the elementary theory of fields.

The Tarski-Mostowski-Robinson theorem shows that, if we have shown that a given theory is decidable and that $\boldsymbol{M}$ is a model of that theory, then the set of natural numbers cannot be defined in the model $\boldsymbol{M}$. For example, if $\boldsymbol{M}$ is the model in the language of arithmetic whose domain is the real numbers, a famous theorem of Tarski proves that the set of sentences true in this model is decidable; consequently, it follows from the Tarski-Mostowski-Robinson theorem that the set of natural numbers cannot be defined in this model $\boldsymbol{M}$. All this is very similar to the argument that shows that $\boldsymbol{Q}$ is undecidable, and, however, $\boldsymbol{Q}$ is so weak that it does not prove several arithmetical properties that we consider to be obvious.

We assume the inductive definitions of addition in terms of successor, and of multiplication in terms of addition and successor, and that the successor symbol defines a 1-1 function whose range is everything but zero. The formal system for first-order arithmetic $\boldsymbol{Q}$ is the following, taking for granted the usual proof machinery of first-order logic (for detail, see [EC00] chapter 21):

Q1 $(x_1' \approx x_2') \rightarrow x_1 \approx x_2$

Q2 $0 \neq x_1'$

Q3 $(x_1 \neq 0) \rightarrow \exists x_2(x_1 \approx x_2')$

Q4 $x_1 + 0 \approx x_1$

Q5 $x_1 + (x_2)' \approx (x_1 + x_2)'$

Q6 $x_1 \cdot 0 \approx 0$

Q7 $x_1 \cdot (x_2)' \approx (x_1 \cdot x_2) + x_1$

Note that **Q1-Q7** are (abbreviations of) wffs, not schemas.

All the theorems of ***Q*** are supposed to true of the natural numbers. We shall assume that ***Q*** has the syntactic property of being consistent : there is no wff A such that both $\vdash_{\boldsymbol{Q}} A$ and $\vdash_{\boldsymbol{Q}} \neg A$.

6 Proving the weakness of the system *Q*

Although the system ***Q*** formalizes a sufficient amount of properties of the natural numbers to be able to represent every recursive function, the system is not strong enough to prove many basic facts of arithmetic. For example, even a simple a wff as $x \neq x'$ cannot be proved in ***Q***. But wow can we demonstrate that? We know how to show a wff is a theorem: exhibit a proof. But how can we show that there is *no* proof?

We shall see here that the idea of "finitely-presented models" in some sense than helps to show how some arithmetic properties fail to be proved in ***Q***. We are not using strictly finite models here, but finitely-presented models in the sense of models described by finite tables.

Suppose we can exhibit something which satisfies all the axioms of the system ***Q***, that is, a model of ***Q***. The rules of proof never lead us from wffs that are true about something to ones that are false, so every theorem of ***Q*** must also be true in that model. Thus all we have to do is show something that satisfies all the axioms of ***Q*** and yet $x \neq x'$ is false in it. Then $x \neq x'$ cannot be a theorem of ***Q***.

To present such a model we need two objects which are not natural numbers. Any two will do; for example, the Moon and the Sun, or beer mugs, as Hilbert used to say, If we label them α and β, the model then consists of the natural numbers supplemented by α and β with the following tables interpreting $'$, $+$, $\cdot$:

$+$	n	α	β
m	$m+n$	β	α
α	α	β	α
β	β	β	α

$'$	sucessor of x
n	$n+1$
α	α
β	β

$\times$	0	$n \neq 0$	α	β
0	0	0	α	β
$m \neq 0$	0	$m \cdot n$	α	β
α	0	β	β	β
β	0	α	α	α

To show that this is a model of ***Q*** we have to assume that the axioms and hence theorems of ***Q*** are true of the natural numbers. Then it's easy to verify that they are also true when we have α and β. But the successor of α is α, and hence $x \neq x'$ cannot be a theorem of ***Q***.

Here is a list of wffs all of which are true of the natural numbers but cannot be proved in ***Q*** (that i,s they are independent of ***Q***). This can be verify using the same model.

Theorem 6.1. *If the theorems of **Q** are true of the natural numbers, then the following are not theorems of **Q**, where x, y, z are distinct variables (parentheses deleted for legibility):*

a $x \neq x'$

b $x + (y + z) \approx (x + y) + z$

c $x + y \approx y + x$

d $0 + x \approx x$

e $\exists x(x' + y \approx z) \rightarrow y \neq z$

f $x \cdot (y \cdot z) \approx (x \cdot y) \cdot z$

g $x \cdot y \approx y \cdot x$

h $x \cdot (y + z) \approx (x \cdot y) + (x \cdot z)$

i $x \cdot 1 \approx x$

Proof. The counter-examples given by the tables above are the following:

a $\alpha' = \alpha$

b $\alpha + (\alpha + \alpha) = \alpha + \beta = \alpha \neq \beta = \beta + \alpha = (\alpha + \alpha) + \alpha$

c $\alpha + \beta = \alpha \neq \beta = \beta + \alpha$

d $0 + \alpha = \beta \neq \alpha$

e $\beta' + \alpha = \beta + \alpha$ and $\alpha' + \beta = \alpha + \beta = \alpha$.
Intuitively, the wff says "not both y is less than z and z is less than y"

f $\alpha \cdot (\alpha \cdot \alpha) = \alpha \cdot \beta = \beta \neq \alpha = \beta \cdot \alpha = (\alpha \cdot \alpha) \cdot \alpha$

g $\alpha \cdot \beta = \beta \neq \alpha = \beta \cdot \alpha$

h $\alpha \cdot (\alpha + \alpha) = \alpha \cdot \beta = \beta \neq \alpha = \beta + \beta = (\alpha \cdot \alpha) + (\alpha \cdot \alpha)$

i $\alpha \cdot 1 = \beta \neq \alpha$

□

It is possible to use finite models instead of these finitely-presented models to show the independence of the above wffs?

Yes, at least in some cases. For example, to show the independence of $x \cdot 0 \approx 0$ from **Q** we may use the following three-valued table:

+	0	1	2
0	0	1	2
1	1	0	2
2	2	2	2

×	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

We do not know if it is possible to give finite models for the independence of all the wffs above; sometimes it us easier to use infinite rather than finite models !

7 Tarski's High School Problem and exotic identities

The high school identities is the set **HSI** of eleven basic identities of the positive integers N with the operations sum (+), product (×) and exponentiation (↑) that everyone learns in high school:

1. $x + y \approx y + x$
2. $x + (y + z) \approx (x + y) + z$
3. $x \times 1 \approx x$
4. $x \times y \approx y \times x$
5. $x \times (y \times z) \approx (x \times y) \times z$
6. $x \times (y + z) \approx (x \times y) + (x \times z)$
7. $1^x \approx 1$
8. $x^1 \approx x$
9. $x^{(y+z} \approx x^y \times x^z$
10. $(x \times y)^z \approx x^z \times y^z$
11. $(x^y)^z \approx x^{y \times z}$.

The subset of the first six identities not involving exponentiation is called $\widehat{\textbf{HSI}}$.

These identities are derived from the natural numbers with the successor operation (see Section 6) and are among the most familiar of the equational theories in mathematics.

Alfred Tarski asked whether **HSI** was a basis for the equational theory of N, a problem known as *Tarski's High School Problem.* A nice exposition of the problem can be fond in [BY])

In 1980, Alex Wilkie (cf. [Wil00]) provided a negative answer to Tarski's High School Problem, obtaining the first *exotic identity* $W(x, y)$,that is, an identity not provable from **HSI**:

$W(x, y) = ((1 + x)^x + ((1 + x + x^2)^x)^y) \times ((1 + x^3)^y + (1 + x^2 + x^4)^y)^x \approx ((1 + x)^y + (1 + x + x^2)^y)^x \times ((1 + x^3)^x + (1 + x^2 + x^4)^x)^y$

Though it may very hard to find such counter-example, a simple proof of this result can be given by the existence of finite models of **HSI** that do not satisfy $W(x, y)$. The first such example was founded by Gurevič (cf. [Gur85]), who gave an extremely complicated algebra with 59 elements (Wilkie's proof was syntactic).

Many smaller counter-models have been found since this first model. Even today it is now known what is the size of smallest counterexample showing the independence of $W(x, y)$, but a model with 12 elements was found (cf. [BY]):

+	**1**	**2**	**3**	**4**	**a**	**b**	**c**	**d**	**e**	**f**	**g**	**h**
1	2	3	4	4	2	3	*d*	3	3	3	3	4
2	3	4	4	4	3	4	3	4	4	4	4	4
3	4	4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4	4	4
a	2	3	4	4	*b*	4	*b*	3	*h*	3	3	4
b	3	4	4	4	4	4	4	4	4	4	4	4
c	*d*	3	4	4	*b*	4	*b*	3	3	3	3	4
d	3	4	4	4	3	4	3	4	4	4	4	4
e	3	4	4	4	*h*	4	3	4	4	3	*h*	4
f	3	4	4	4	3	4	3	4	3	4	3	4
g	3	4	4	4	3	4	3	4	*h*	3	4	4
g	4	4	4	4	4	4	4	4	4	4	4	4

$\wedge$	**1**	**2**	**3**	**4**	**a**	**b**	**c**	**d**	**e**	**f**	**g**	**h**
1	1	2	3	4	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*
2	2	4	4	4	*b*	4	*b*	4	4	4	4	4
3	3	4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4	4	4
a	*a*	*b*	4	4	*c*	*b*	*c*	*b*	*h*	4	4	4
b	*b*	4	4	4	*b*	4	*b*	4	4	4	4	4
c	*c*	*b*	4	4	*c*	*b*	*c*	*b*	4	4	4	4
d	*d*	4	4	4	*b*	4	*b*	4	4	4	4	4
e	*e*	4	4	4	*h*	4	4	4	4	4	*h*	4
f	*f*	4	4	4	4	4	4	4	4	4	4	4
g	*g*	4	4	4	4	4	4	4	*h*	4	4	4
h	*h*	4	4	4	4	4	4	4	4	4	4	4

$*$	**1**	**2**	**3**	**4**	**a**	**b**	**c**	**d**	**e**	**f**	**g**	**h**
1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	4	4	4	4	4	4	4	*f*	4	4	4
3	3	4	4	4	*e*	4	4	4	*g*	4	*e*	*h*
4	4	4	4	4	4	4	4	4	4	4	4	4
a	*a*	*c*	*c*	*c*	*c*	*c*	*c*	*c*	*c*	*c*	*c*	*c*
b	*b*	4	4	4	4	4	4	4	4	4	4	4
c	*c*	*c*	*c*	*c*	*c*	*c*	*c*	*c*	*c*	*c*	*c*	*c*
d	*d*	4	4	4	*f*	4	4	4	4	4	4	4
e	*e*	4	4	4	4	4	4	4	*h*	4	4	4
f	*f*	4	4	4	4	4	4	4	4	4	4	4
g	*g*	4	4	4	*h*	4	4	4	4	4	*h*	4
h	*h*	4	4	4	4	4	4	4	4	4	4	4

After Wilkie's discovery of his exotic identity there was an effort to find smaller identities, but nobody knows which is the simplest one. Also, it is conjectured that this model with 12 elements is minimal: the lower bound is 8, as shown in [Jac96]. There is still a lot of work to settle the smallest exotic identity and the exact cardinality of the minimal model.

8 The Finite Basis Problem and weird small models

An *algebra* (or an *algebraic structure*) is a mathematical object
$\mathcal{A} =< A, o_1, \ldots, ø_n, R_1, \ldots, R_m >$
consisting of a non-empty set A (the universe of the algebra) together with a finite collection of operations $o_1, \ldots, ø_n$ and relations $R`1, \ldots, R_m$ defined on the set A. When the the universe A of the algebra is finite we say it is a *finite algebra*. When the operations are zero-ary, the algebras has constants (as it happens with Boolean algebras). When there is no risk of confusion, the algebra $\mathcal{A}$ is sometimes denoted by its universe A.

An algebra my be defined by requiring that it satisfies certain axioms, or by specifying the operations and relations. This is specially interesting in the case of small finite algebras, and is done by means of the *Cayley tables*.

A Cayley table is like the familiar tables for multiplication or addition, but generalized to arbitrary algebraic operations. Cayley tables are extremely useful in the description of small finite algebras that have binary operations; almost all many-valued logics in the literature use Cayley tables to specify the interpretation of logical connectives. So the study of many-valued logics is a sense part of (or a complement of) the study of finite algebras.

Examples of algebras are groups, rings, fields, vector spaces, Boolean algebras, etc. In these cases the algebras are defined by requiting that they satisfy specific axioms.

A *groupoid* is an algebraic structure on a set closed under a binary operator $\star$ (i.e., applying the binary operator to two elements of a given set S returns a value in S). No other property such as associativity, commutativity, etc., are required in a groupoid.

A*quasigroup* is a groupoid S such that for all $a, b \in S$ there exist unique $x, y \in S$ such that: $a \star x = b$ and $y \star a = b$

No other restrictions are posed; thus a quasigroup need not have an identity element, not be associative, etc. Quasigroups are precisely groupoids whose multiplication tables are *Latin squares*: in each row and each column of the table each element of S occurs exactly once.

For example:

$\star$	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

There are 12 Latin squares of order 3, and only two Latin squares of order 2:

$\dot{\vec{\vee}}$	0	1
0	1	0
1	0	1

$\equiv$	0	1
0	0	1
1	1	0

that correspond to *exclusive disjunction* $\vec{\vee}$·and *equivalence* ≡in binary (classical) logic reading 0 for *false*and 1 for *true*.

A *semigroup* is an associative groupoid, i.e., a grupoid where the binary operator in which the multiplication operation is associative. No other restrictions are placed on a semigroup; thus a semigroup need not have an identity element and its elements need not have inverses within the semigroup. A semigroup with an identity is called a *monoid*.

For people with interest in category theory, it is worth noting that monoids can be viewed essentially as categories with a single object, since the axioms required for a monoid operation are exactly those required for morphism composition when restricted to endomorphisms (i.e., to the set of all morphisms starting and ending at a given object).

For example, the following is a semigroup (which happens to be also commutative):

⋆	0	1	2
0	1	1	0
1	1	1	1
2	0	1	2

It is easy to check that the equations $x\star(y\star z) \approx (x\star y)\star z$ (and also that $x\star y \approx y\star x$) hold for all elements x, y and z.

Equations that hold identically (such as the above) on an algebra are said to be *satisfied* by the algebra.

Given a set of identities S, a *basis* (or an *axiomatic basis of identities*) for the identities is a subset of S_0 of S from which all the identities in S can be derived (by means of the usual ways that equations derive other equations; this has a technical meaning, however, that is explained in universal algebra). If S_0 is finite, we say that the identities of A are *finitely based* (or *finitely axiomatizable*).

A finite algebra is said to be *inherently non-finitely based* if it is not finitely based, and any algebra that contains it is also not finitely based (this can be rephrased in the language of universal algebra by the fact that any locally finite variety containing it is not finitely based).

Tarski's Finite Basis Problem is the general question, posed by him around 1960, of deciding whether an arbitrary finite algebra can be finitely axiomatizable. Ralph McKenzie solved this hard problem in 1995(cf. [McK96]) showing that there is no algorithm for determining which finite algebras (defined by finitely many fundamental operations) ate finitely axiomatizable. So, in the absence of an algorithm, the problem has to be dealt in a case-by-case basis. For example, it is known that every finite group, every finite ring and every commutative semigroup is finitely axiomatizable (or finitely based).

The first example of an finite algebra whose identities cannot be finitely axiomatizable was a 7 element groupoid found by Lyndon in 1954:

⋆	**0**	**1**	**2**	**3**	**4**	**5**	**6**
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0
4	0	4	5	6	0	0	0
5	0	5	5	5	0	0	0
6	0	6	6	6	0	0	0

Another example of a finite semigroup whose identities are not finitely axiomatizable we given by Perkins, in 1968. This semigroup is inherently non-finitely based:

∗	**0**	**1**	**2**	**3**	**4**	**5**
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	0	4	0	2
3	0	3	5	0	3	0
4	0	4	2	0	4	0
5	0	5	0	3	0	5

In 1965 Murskii (cf. [Mur65]) discovered a 3 element groupoid. whose identities are be finitely axiomatizable and which is inherently non-finitely based:

⋆	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

It is a very instructive exercise to show that there are exactly five two-element models of **HSI**. In principle, there are $(2^4)^3 = 2^{12}$ such models, but playing with the equations it is not difficult to see that they reduce to the following:

Model I

+	1	0
1	1	1
0	1	0

×	1	1
1	1	0
0	0	0

↑	1	1
1	1	1
0	0	1

Model II

+	1	0
1	1	1
0	1	0

×	1	1
1	1	0
0	0	0

↑	1	1
1	1	1
0	0	0

Model III

+	1	0
1	1	0
0	0	0

×	1	1
1	1	0
0	0	0

↑	1	1
1	1	1
0	0	0

Model IV

+	1	0
1	0	0
0	0	0

×	1	1
1	1	0
0	0	0

↑	1	1
1	1	1
0	0	0

Model V

+	1	0
1	0	1
0	1	0

×	1	1
1	1	0
0	0	0

↑	1	1
1	1	1
0	0	0

It is not an easy problem to find finite models for **HSI** in general (in the present case we are using elementary counting methods). A proof that all **HSI** identities are valid in every 2-element **HSI**-algebra was only obtained in 2004 (cf. [Asa04]).

8.1 Relevant Internet sites

Much more on this topic an be found in the webpages maintained by Marcel Jackson at:
http://www.maths.utas.edu.au/People/Jackson/cayley.html
and by Stanley N. Burris at:
http://www.thoralf.uwaterloo.ca/
The quasigroup completion problem concerns completing a a partially filled table in order to obtain a complete Latin square (or a proper quasigroup multiplication table). This is a difficult computational probme (NP-complete.

Carla Gomes maintains a webpage at:
http://www.cs.cornell.edu/Info/People/gomes/QUASIdemo.html
that demonstrates the quasigroup completion problem for 10×10 Latin squares, using color instead of numbers.

Jaroslav Jezek's webpage has a program that checks whether a groupoid satisfies a set of equations, and whether a groupoid is subdirectly irreducible:
http://adela.karlin.mff.cuni.cz/ jezek/

9 Why modal logics are not many-valued

A *normal propositional modal logic*is any extension of the classical propositional calculus that contains, besides all classical propositional theorems, the formula:

(**K**) $\Box(p \to q) \to (\Box p \to \Box q)$

and is closed under the following rules:

Modus Ponens;

Uniform substitution: if $\vdash \alpha$, p is a variable of α and β is any formula, then $\vdash \alpha[\beta/p]$ (where $\alpha[\beta/p]$ denotes the substitution of all variables p by β);

Necessitation (Nec): if $\vdash \alpha$ then $\vdash \Box\alpha$

The smallest normal modal logic is called **K**. The usual extensions of **K** are defined by adding the following axioms, where $\diamond p$ is defined as $\neg\Box\neg p$:

($\mathbf{D}$): $\Box p \to \diamond p$, defining the system **KD**;

($\mathbf{T}$): $\Box p \to p$, defining the system **KT** or simply **T**;

($\mathbf{B}$): $p \to \Box \diamond p$, defining the system **B**;

($\mathbf{4}$): $\Box p \to \Box\Box p$, defining the system **S4**;

($\mathbf{5}$): $\diamond p \to \Box\diamond$, defining the system **S5**.

A well-known theorem of 1940 due to James Dugundji proves that it is not possible to characterize modal logics by means of finite matrices. This shows the necessity of a new kind of semantics; Saul Kripke was the first to propose semantics for modal logics, based on the idea of possible words (known today as *Kripke's semantics* or *possible worlds semantics*); we do not treat the possible-worlds semantics, as our purpose here is to study the capability and limitations of finite models. For detailed expositions of this semantics, we suggest [HC96] or [CP01] if you read Italian.

Clarence Lewis proposed in 1918 the first formal modal logics, although the idea of reasoning with modalities was as old as Aristotle, and much discussed in medieval logic. In 1932 C. Lewis and Langford publish their book *Symbolic Logic*, now a classic. Part of their motivation was to avoid the paradoxes of material implication:

- $\alpha \to (\neg\alpha \to \beta)$
- $\beta \to (\alpha \to \beta)$
- $(\alpha \to \beta) \vee (\beta \to \alpha)$

The authors introduced the notion of *strict implication* ($\multimap$) to solve such problems, in such a way that the followings formulas are not valid:

- $\alpha \multimap (\neg\alpha \multimap \beta)$
- $\beta \multimap (\alpha \multimap \beta)$

This did not solve all problems, since for example the formula $(\alpha \wedge \neg\alpha) \multimap \beta$ would still be valid. They introduced the modal systems **S4** and **S5** (along with certain weaker systems **S1**, **S2**and **S3**) but did not know how to semantically characterize such systems. They believed that it was possible to characterize these systems via multi-valued semantics and tried several possibilities without success, till Dugundji's result proved that this was impossible.

There exists however another tradition in logic, due to Jan Łukasiewics, which proposes a "multi-valued reading" to the modal notions of Aristotle.

Why is it so difficult to give a semantic interpretation to the modal operator ($\Box$)? The problem is that the value of $\Box\alpha$ *not* depends in a complicated way upon the value of α. Consider *alpha* as "Every dog is a dog"; then obviously $\Box\alpha$ is true. But if α is "Every dog is a pet", it seems clear that $\Box\alpha$ is false. It is had problem for formal semantics to distinguish between these subleties.

To demonstrate his theorem, Dugundji followed Gödel's idea discussed above, which shown showing that the intuitionistic logic has no finite-valued semantics. Later on, Gödel proved that intuitionistic logic is equivalent (inter-translatable) to the modal logic **S4**.

The argument by Dugundji consists of the following points:

(a) He shows that for each matrix with n values, there exists a (modal) disjunction with $n + 1$ variables (call them Dugundji's formulas) which takes distinguished values;

(b) He then shows that there exists an infinite matrix that assigns distinguished values to every formula of **S5**;

(d) Moreover, this infinite matrix falsifies all Dugundji's formulas);

(d) From (a), (b)and (c) it follows that no modal system $S \subseteq$ **S5** can be characterized by matrices with finite truth-values.

Definition 9.1. *A matrix $\mathscr{M}$ is a triple $\mathscr{M} = \langle M, D, O\rangle$, where:*
- $M \neq \emptyset$;
- $D \subseteq M$ are the distinguished values;
- O is a set of operation.
It is clear that $\mathscr{M}$ is an algebra.

Definition 9.2. *A matrix characterizes a logic system* **S** *if every theorem of* **S** *(and only them) receive distinguished values.*

Fact 1 Consider the following formulas, where $(x \equiv y) =_{Def} \Box((x \to y) \wedge (y \to x))$:

F1 $p \equiv q$, written in the variables p and q;
F2 $(p \equiv q) \vee (p \equiv r) \vee (q \equiv r)$, written in the variables p, q and r;
F3 $(p \equiv q) \vee (p \equiv r) \vee (p \equiv s) \vee ... \vee (r \equiv s)$, written in the variables p, q, r and s;
. . .
Fn $(\frac{n(n+1)}{2})$ disjunctions in $p_1, \cdots, p_{n+1}$ variables.

Given a matrix with n values and a formula **Fn** having $n+1$ variables, the formula $\Box(p \leftrightarrow p)$, which is a tautology in the matrix, will appear among the disjuncts. Since **Fn** is itself a disjunction, it will be true in this matrix.

Fact 2 Consider the following infinite matrix $\mathscr{M} = \langle M, D, O\rangle$, where:

- the set of values is $M = \wp(\mathbb{N})$;

- the set of distinguished values is $D = \{\mathbb{N}\}$;

- the operations are $O = \{\cap, \cup, ^{-}, \blacksquare\}$, such that $\cap, \cup, ^{-}$ are the usual set operations of conjunction, disjunction and complement, and

$$\blacksquare X = \begin{cases} \mathbb{N} & \text{if } \blacksquare X = \mathbb{N} \\ \emptyset & \text{other case.} \end{cases}$$

Let $v : Prop \longrightarrow M = \{A, B, C, \ldots\}$ be an assignment of elements of M to the propositional variables; this function can be extend to all formulas in the following way:

- $v(\bot) = \emptyset$;
- $v(X \wedge Y) = v(X) \cap v(Y)$;
- $v(X \vee Y) = v(X) \cup v(Y)$;
- $v(\neg X) = \overline{v(X)}$;
- $v(\Box X) = \blacksquare(v(X))$.

Consequently, we have: $v(X \to Y) = v(\neg X \vee Y) = \overline{v(A)} \cup v(B)$.

It is easy to proof that $\mathscr{M}$ satisfies the axioms of **S5** and that the rules preserve validity; this means that $\mathscr{M}$ is a model for **S5** (i.e., **S5** is sound with respect to $\mathscr{M}$) .

Exercise: Show that $v(\Box p \to p) = \mathbb{N}$, for any $v(p)$.

Theorem 9.3. *No characteristic matrix for a subsystem of* **S5** *can have a finite number of truth-values.*

Proof. We prove that no Dugundji's formula **Fn** can receive a distinguished value in the matrix $\mathscr{M}$ that satisfies the thesis of **S5**.

Take the following valuation v that assigns singleton $\{k\}$ to the propositional variable p_k. We know that $v(p \equiv q) = \blacksquare(\overline{P} \cup Q) \cap \blacksquare(P \cup \overline{Q})$, where $v(p) = P$, $v(q) = Q$.

Note that if P and Q are singletons, then $\overline{P} \neq \mathbb{N}$ and $\overline{Q} \neq \mathbb{N}$. Moreover, $P \subseteq \overline{Q}$ and $Q \subseteq \overline{P}$, hence $v(p \equiv q) = \blacksquare(\overline{P}) \cup \blacksquare(\overline{Q}) = \emptyset$, consequently, all Dugundji's formulas take value $\emptyset$ in this matrix, i.e., $v(Fn) = \emptyset$.

Therefore no Dugundji's formula takes a distinguished value in the infinite matrix that satisfies **S5**, and hence these formulas cannot be theorems of **S5**. However, for each finitary matrix exists some Dugundji's formula that is a tautology in this finitary matrix. Therefore, this finitary matrix cannot characterize **S5**.

To show that it cannot characterize a subsystem **S** of **S5**, suppose that **S** can be characterized by a finite matrix with n truth values. This finite matrix will satisfy Dugundji's formula **Fn+1**, and hence, as this matrix by hypothesis characterizes **S**, **Fn+1** would be a theorem of **S**, and consequently a theorem of **S5**, which is an absurd. □

To conclude, this theorem shows that any finitary matrix is at most correct, but never complete for modal logics that are subsystems of **S5**.

10 Why intuitionistic logic is not many-valued

Intuitionistic logic basically criticizes the Aristotelian law of excluded middle:

(exc) $\alpha \vee \neg\alpha$

but is happy with the law of explosion:

(exp) $\alpha \to (\neg\alpha \to \beta)$

).

An axiomatic system for intuitionistic propositional logic **IPL** is obtained precisely by dropping (Ax9) ($\alpha \vee (\alpha \to \beta)$) and adding (exp) to the axioms $\mathbf{CPL}^+$. This results in a very significant departure from classical reasoning.

The founder of Intuitionism, Alois Brouwer, argued that (exc) is clearly valid in finite situations, but its extension to statements about infinite collections would cause problems in logical reasoning.

In a certain sense intuitionistic logic replaces truth for "justification " in its logical bases. Instead of a two-valued truth assignment scheme, it accepts a kind of third, indeterminate truth-value. A proposition may be provably justified, or provably not justified, or undetermined.

It seems obvious that intuitionistic logic would be incomplete if we maintain the classical two=valued semantics. On the light of this interpretation via a third indeterminate truth-value, one may be tempted to think on the possibility of characterizing **IPL** as a three-valued logic, or at least as some finite-valued logic. This is, however impossible, as shown by Gödel in [G3̈2].

We do not show details here, and prefer to concentrate on less known arguments about impossibility of characterization by finite-valued matrices, as it is the case of modal logic and paraconsistent logics (although Gödel's arguments have inspired all those).

It is relevant here to recall that he has used an infinite sequence of finite-valued logics that are known today as *Gödel's many-valued logics* G_n, and that constitute a family of intermediate logics between classical and intuitionistic. The truth-tables of G_n are given by the following matrices for interpreting the signature Σ , where the truth-values for each G_n are $V_n = \{0, 1, \ldots, n-1\}$, and 0 is the only distinguished value:

$$
\begin{aligned}
v(\alpha \wedge \beta) &= max\{v(\alpha), v(\beta)\};\\
v(\alpha \vee \beta) &= min\{v(\alpha), v(\beta)\};\\
v(\alpha \to \beta) &= 0, \text{ if } v(\alpha) \geq v(\alpha);\\
&\quad v(\beta) \text{ otherwise};\\
v(\neg\alpha) &= 0, \text{ if } v(\alpha) = n-1;\\
&\quad n-1 \text{ otherwise}
\end{aligned}
$$

Some years after Gödel's result about the impossibility of characterizing **IPL** as finite-valued, S. Jaśkowski in [Jas36] proved that IL can be characterized by means of an infinite class of finite matrices.

Saul Kripke has proposed a new very interesting semantics known as *possible-worlds semantics* for which intuitionistic logic can be proven to be sound and complete.

Gödel also found a translation from classical logic into intuitionistic logic without $\wedge$ (and without $\exists$ for the first-order case), showing that a formula $\alpha$is classically provable iff its translation $g(\alpha)$ provable intuitionistically provable. In this way, if a contradiction were classically provable, it would also be intuitionistically provable; this was used in [G3̈3] to show that intuitionistic arithmetic is as consistent as classical arithmetic (i.e., both are equiconsistent). Based on this, and in h fact that intuitionistic logic distinguishes formulas which are classically indistinguishible (i.e., equivalent) he thought that intuitionistic logic is richer than classical logic. What is interesting form ur poitn of view is that, *mutatis mutandis*, the same argument is able to show that paraconsistent logic is also richer than classical logic.

11 Why certain paraconsistent logics are not many-valued

It is well-known (cf. [Men97]), any logic having (Ax1) and (Ax2) as axioms, and *modus ponens* (MP) as the only inference rule enjoys the following *Deduction Theorem*:[5]

(DT) $\Gamma, \alpha \vdash \beta \;\Leftrightarrow\; \Gamma \vdash \alpha \to \beta$

The following Theorem will be tacitly used throughout this chapter:

Theorem 11.1. (1) Any axiomatic extension of positive classical logic respects (DM); (2) There are extensions-by-rules of positive classical logic that do not respect (DM).

Any implication enjoying (DT) and (MP) will from now be called *deductive*.

It should be noted, as proven in [BT72], that the problem of determining whether or not the deduction theorem holds for an arbitrarily given partial implicational calculus (and hence for an arbitrarily given generalized partial propositional calculus) is recursively unsolvable.

We will consider for this section a signature Σ containing the binary connectives $\wedge$, $\vee$, $\to$, and the unary connective $\neg$, and atomic formulas $\mathcal{P} = \{p_n \; : \; n \in \omega\}$; *For*will denote the set of formulas generated by $\mathcal{P}$ over Σ. Σ° will denote the signature obtained by the addition of a new unary connective $\circ$ to the signature Σ, and For° will denote the formulas for the signature Σ°.

We first consider the simple, yet very interesting, three-valued logic *Pac* studied in [Avr91]. The same logic had already appeared in [Avr86] under the name $RM_3^{\supset}$, and, before that, in [Bat80], where under the name of PI^s.

Pac is given by the following matrices:

$\wedge$	2	1	0
2	2	1	0
1	1	1	0
0	0	0	0

$\vee$	2	1	0
2	2	2	2
1	2	1	1
0	2	1	0

$\to$	2	1	0
2	2	1	0
1	2	1	0
0	2	2	2

	$\neg$
2	0
1	1
0	2

where both 1 and 2 are designated values.

In *Pac* there is no formula α such that $\alpha, \neg\alpha \vdash_{Pac} \beta$ for all β, thus it is a paraconsistent logic.

A *classical negation* can be added to *Pac* by adding a connective interpreted by the following matrix :

	$\sim$
2	0
1	0
0	2

[5] This is not always true, though, for logics extending (Ax1), (Ax2) and (MP) by the addition of new inference rules.

It is clear that such a negation cannot be definable from other connectives of *Pac*, because any truth-function of *Pac* is closed within $\{\frac{1}{2}\}$, that is, taking this value as input will also have $\frac{1}{2}$ as output.

In adding to *Pac* either a supplementing negation as above or a bottom particle, we will obtain a well-known three-valued conservative extension of it, called $\mathbf{J}_3$. This logic has also appeared several times independently in the literature (see [CCM], Section 2).

The expressive power of this logic was studied in [Avr99] and in [CMdA00]. The latter paper, renaming it as **LFI1**, explores the possibility of applying this logic to databases.

we will see that three-valued logic **LFI1** (or $\mathbf{J}_3$) is axiomatizable and sound and complete with respect to the truth-tables given above (cf. [CMdA00]).

However, many extensions of *Pac* given by axiomatic definitions will not have *any* finite-valued semantics. We will show here in all details, closely following [CCM].

PI The paraconsistent logic *PI* is obtained[6] from $\mathbf{CPL}^+$ in the signature Σ) by adding the axiom schema of 'excluded middle':

(exc) $\alpha \vee \neg\alpha$.

mbC The logic **mbC** is obtained from *PI* in the extended signature Σ°, by adding the axiom schema:

(bc1) $\circ\alpha \rightarrow (\alpha \rightarrow (\neg\alpha \rightarrow \beta))$.

mCi The logic **mCi** is obtained from **mbC** by the addition axiom schemas:

(ci) $\neg\circ\alpha \rightarrow (\alpha \wedge \neg\alpha)$

(co)$_n$ $\circ\neg^n\circ\alpha$ $\quad (n \geq 0)$.

bC The logic **bC** is obtained from **mbC** by adding the axiom schema:

(cf) $\neg\neg\alpha \rightarrow \alpha$.

Ci The logic **Ci** is obtained from **mCi** by adding the axiom schema (cf).

mbCe The logic **mbCe** is obtained from **mbC** by adding axiom schema:

(ce) $\alpha \rightarrow \neg\neg\alpha$.

mCie The logic **mCie** is obtained from **mCi** by adding axiom (ce).

bCe The logic **bCe** is obtained from **bC** by adding axiom (ce).

Cie The logic **Cie** is obtained from **Ci** by adding axiom (ce).

We show here, following the results given in [CCM] (Sections 3.3 and 4.4) that all these paraconsistent logics cannot be characterized by truth-functional finite valued matrices. This implies that we will have to use more sophisticated semantics. The technique used here is inspired by the arguments of K. Göodel and J. Dugundji about finite uncharacterizability of intuitionistic and normal modal logics, respectively, obtains sufficient conditions for the paraconsistent case.

[6] Introduced in [Bat80].

Theorem 11.2. Consider the following infinite-valued matrix over Σ° whose truth-values are the ordinals in $\omega + 1 = \omega \cup \{\omega\}$, of which all elements in ω are designated. The connectives of Σ° are interpreted by mappings $v : For^\circ \longrightarrow \omega + 1$ such that:

$$
\begin{aligned}
v(\alpha \wedge \beta) &= 0, \text{ if } v(\beta) = v(\alpha) + 1 \text{ or } v(\alpha) = v(\beta) + 1; \\
&\ max(v(\alpha), v(\beta)), \text{ otherwise;} \\
v(\alpha \vee \beta) &= min(v(\alpha), v(\beta)); \\
v(\alpha \to \beta) &= \omega, \text{ if } v(\alpha) \in \omega \text{ and } v(\beta) = \omega; \\
&\ v(\beta), \text{ if } v(\alpha) = \omega \text{ and } v(\beta) \in \omega; \\
&\ 0, \text{ if } v(\alpha) = \omega \text{ and } v(\beta) = \omega; \\
&\ max(v(\alpha), v(\beta)), \text{ otherwise;} \\
v(\neg\alpha) &= \omega, \text{ if } v(\alpha) = 0; \\
&\ 0, \text{ if } v(\alpha) = \omega; \\
&\ v(\alpha) + 1, \text{ otherwise;} \\
v(\circ\alpha) &= \omega, \text{ if } 0 < v(\alpha) < \omega; \\
&\ 0, \text{ otherwise.}
\end{aligned}
$$

Let **L** be any logic with a negation and a deductive implication defined over Σ° and extending $\mathbf{CPL}^+$ which is sound with respect to the infinite-valued matrices defined above. Then **L** cannot be semantically characterized by finite-valued matrices.

Proof. Let **L** be any logic satisfying the given hypothesis. Define the following formulas over Σ°:

$$\varphi_{ij} \stackrel{\text{def}}{=} \circ p_i \wedge p_i \wedge \neg p_j, \text{ for } 0 \le i < j;$$

$$\psi_n \stackrel{\text{def}}{=} \bigvee_{0 \le i < j \le n} (\varphi_{ij} \to p_{n+1}), \text{ for } n > 0.$$

It is easy to see that all formulas ψ_n can be assigned the non-designated truth-value ω in the infinite model according to the given matrices: Just assign $v(p_i) = i$ for $0 \le i \le n$ and $v(p_{n+1}) = \omega$. On the other hand, the formula ψ_n must be a tautology in every m-valued set of matrices that is sound for **L** and such that $m < n$. Indeed, if $m < n$, by the Pigeonhole Principle of elementary combinatorics there exist i and j such that $v(p_i) = v(p_j)$. But the formula $(\circ\alpha \wedge \alpha \wedge \neg\alpha) \to \beta$ is a theorem of **L** for every α and β, then it must be evaluated as designated in every set of adequate matrices for **L**. The same occurs, of course, with the formula $\alpha \to (\alpha \vee \beta)$. It follows from the validity of such formulas (and by *modus ponens*) that ψ_n is also validated in every adequate set of m-valued matrices which is sound for **L** and such that $m < n$, although, as mentioned before, no formula ψ_n can be a theorem of **L**. Therefore, **L** cannot be semantically characterized by any collection of finite-valued matrices. □

Theorem 11.3. Consider next the following infinite-valued matrix over Σ° whose truth-values are the ordinals in ω, of which 0 is the only non-designated value. The connectives of Σ° are now interpreted using maps $v : For^\circ \longrightarrow \omega$ such that:

$$
\begin{aligned}
v(\alpha \wedge \beta) &= 1, \text{ if } v(\alpha) > 0 \text{ and } v(\beta) > 0; \\
&\ 0, \text{ otherwise;} \\
v(\alpha \vee \beta) &= 1, \text{ if } v(\alpha) > 0 \text{ or } v(\beta) > 0; \\
&\ 0, \text{ otherwise;} \\
v(\alpha \to \beta) &= 0, \text{ if } v(\alpha) > 0 \text{ and } v(\beta) = 0; \\
&\ 1, \text{ otherwise;}
\end{aligned}
$$

$$
\begin{array}{rcl}
v(\neg\alpha) & = & 1,\ \text{if } v(\alpha) = 0; \\
 & & v(\alpha) - 1,\ \text{otherwise}; \\
v(\circ\alpha) & = & 0,\ \text{if } v(\alpha) > 1; \\
 & & 1,\ \text{otherwise};
\end{array}
$$

Let **L** be any logic with a negation and a deductive implication defined over Σ° and extending **CPL**$^+$ which is sound with respect to the infinite-valued matrices defined above. Then **L** cannot be semantically characterized by finite-valued matrices.

Proof. Let **L** be any logic satisfying the given hypothesis, and let $\neg^i$, for $i \geq 0$, denote i iterations of the negation $\neg$. Define the following formulas over Σ°:

$$\varphi_{ij} \stackrel{\text{def}}{=} \neg^i p \leftrightarrow \neg^j p,\ \text{for } 0 \leq i < j.$$

It is easy to see that the above matrices invalidate all formulas φ_{ij}, assigning to them the non-designated truth-value 0: Just assign $v(p) = j$, and notice that $v(\neg^j p) = 0$ while $v(\neg^i p) > 0$. On the other hand, by the Pigeonhole Principle, any m-valued set of matrices that is sound and complete for **L** will be such that, given some i, there is some $i < j \leq (i + n^n)$, such that $v(\neg^i p) = v(\neg^j p)$, for all v. In that case, φ_{ij} is validated, an absurd. Therefore, **L** cannot be semantically characterized by any collection of finite-valued matrices. □

As we shall see, the above results will help to show that many paraconsistent logics are not finite-valued. Either of them, in fact, allows us to prove, at this point:

Corollary 11.4. The parfaconsistent logics *PI*, **mbC**, **mCi**, **bC**, **Ci**, **mbCe**, **mCie bCe** and **Cie** are not characterizable by finite matrices.

Proof. For the logics *PI* and **mbC**, it is immediate to check that it is sound with respect to both the matrices of Theorem 11.2 and those of Theorem 11.3. To check the result for **bCe** and **Cie**, make use of Theorem 11.2 again. For the other logics, use either the same theorem or Theorem 11.3. □

So we have seen that, departing from the three-valued logic *Pac* and adding appropriate axioms, we obtained a collection of purely infinitely-valued logics.

However, it is interesting to observe that the effect of adding new axioms exhibits a "phase transition–shift" from cases where a new axiom maintains the uncharacterizability by finite matrices to those new axioms surpasses a certain threshold. This will be precisely the case when we add certain axioms to **Ci** for example the resulting logic is not only again characterized by finite matrices, but even by three-valued and two-valued matrices!

LFI1 The logic **LFI1** is obtained by adding to **Ci** the axiom schemas:

- **(ce)** $\alpha \to \neg\neg\alpha$
- **(cj1)** $\bullet(\alpha \land \beta) \leftrightarrow ((\bullet\alpha \land \beta) \lor (\bullet\beta \land \alpha))$;
- **(cj2)** $\bullet(\alpha \lor \beta) \leftrightarrow ((\bullet\alpha \land \neg\beta) \lor (\bullet\beta \land \neg\alpha))$;
- **(cj3)** $\bullet(\alpha \to \beta) \leftrightarrow (\alpha \land \bullet\beta)$.

Thus the logic **LFI1** is an extension of *PI* by several axioms, and is sound and complete with respect to three-valued matrices. Nonetheless, by adding just one further axiom to *PI* we obtain Classical Propositional Logic, which is of course two-valued:

CPL Classical propositional logic **CPL** is an extension of *PI* in the signature Σ, obliging $\neg$ to follow the 'explosion law':

(exp) $\alpha \to (\neg\alpha \to \beta)$

It is clear that $\mathbf{CPL}^+$ has a two-valued semantics (given by the classical tables that interpret classical conjunction, disjunction and implication). So we have seen that departing from two-valued semantics and adding appropriate axioms, we fly high into infinitely-valued semantics, fall down to three-valued, and finally back again to two-valued.

12 Possible-translations semantics

What can we do when the logics are not characterized by finite models? It is possible, in many cases, to combine finite models in as specific so as to semantically characterize such logics, by means of a new semantical concept known as *possible-translations semantics*. Possible-translations semantics were first introduced in 1990 in [Car90b],with emphasis to the case of finite-valued factors. More general treatments of possible-translations semantics can be found in [Car00]) and [Mar99] .

Possible-translations semantics can be used to assign combinations of finite-valued semantics (indeed, in most cases just combinations of three-valued logics) to several paraconsistent logics as *PI*, **mbC**, **mCi**, **bC**, **Ci**, **bCe**, **Cie**, etc., (see [CCM] and [Mar04]). Truth-functional finite-valued logics can also be split in terms of 2-valued logics (fragments of classical logic) as shown in [CCCM03]. According to this idea, copies of classical logic can be combined to give semantics to modal logics.

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